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ANALYSIS OF TWISTED FREE-CONVECTIVE FLOWS INDUCED BY HEAT SOURCES

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Results of numerical simulation of the development of a laminar twisted axisymmetric free-convective jet above a point heat source are presented. Specific features of velocity and temperature profiles as a function of the Prandtl number are presented. It is found that their interaction has a nonlinear character. Detailed tables of numerical solutions are given. Global extrema of the problem are determined.

Introduction. Free-convective flows induced by heat sources are of much interest in engineering since they occur frequently in industry, technological processes, and nature. Thus, the constant attention of specialists to the study of this problem, which is expressed in the appearance of a great number of computational-experimental papers, is understandable (the state-of-the-art and a review of the literature in this field are presented in [1]). However, whereas the specific features of the velocity and temperature fields in non-twisted free-convective jets have been studed rather thoroughly, the characteristics of heat transfer in the presence of flow twisting have received comparatively little attention [2-5]. Information about the structure of these flows bears, for example, a direct relation to the design of some power units, in which elements rotating under unfavorable thermal conditions must often be cooled to maintain an economically acceptable lifetime for the units.

This paper presents results of a comprehensive numerical study of the hydrodynamics and heat transfer in a twisted axisymmetric free-convective jet within the framework of the model of a laminar boundary layer in the Boussinesq approximation at different Prandtl numbers ($0.01 \le \mathrm{Pr} \le 100$).

Basic Equations. We consider a twisted axisymmetric flow induced by a point heat source of intensity Q_0 that rotates with a constant angular velocity. We use a cylindrical system of coordinates x, y, φ with the corresponding velocity components u, v, w. The origin of the system coincides with the position of the point source, and the x axis is directed vertically upward. All properties of the liquid, except for the density, are taken to be constant. Then the basic equations describing the vertical jet flow are wrtitten as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\nu}{y} \frac{\partial}{\partial y} \left(y \frac{\partial u}{\partial y} \right) + g\beta (T - T_\infty) ,$$

$$\frac{\partial}{\partial x} (yu) + \frac{\partial}{\partial y} (yv) = 0 ,$$

$$\frac{w^2}{y} = \frac{1}{\rho} \frac{\partial P}{\partial y} , \tag{1}$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + \frac{vw}{y} = \nu \left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{y} \frac{\partial w}{\partial y} - \frac{w}{y^2} \right) ,$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{\mathrm{Pr}} \frac{\partial}{\partial y} \left(y \frac{\partial T}{\partial y} \right)$$

with the boundary conditions

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$$y = 0: \quad v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = w = 0\,,$$
$$y \to \infty: \quad u \to 0\,, \quad T \to T_\infty\,, \quad w \to 0\,, \quad P \to P_\infty\,. \tag{2}$$

The law of energy conservation requires that at any $x > 0$ the energy transferred by convection be equal to the energy Q_0 liberated by the point source:

$$Q_0 = 2\pi \int_0^\infty \rho C_p u \Delta T y dy = \text{const}\,. \tag{3}$$

Now, two different problems may be considered within the framework of Eqs. (1)-(3). In the first case it is assumed that the circulation of the azimuthal velocity component Γ ($\Gamma = yw$) is equal to zero at the outer boundary of the jet ($y \to \infty$: $\Gamma \to 0$). Then, along with equality (3) one more integral condition appears, namely, the law of angular-momentum conservation for the flow:

$$L_0 = 2\pi \int_0^\infty \rho u w y^2 dy = \text{const}\,. \tag{4}$$

The second case is characterized by the fact that the circulation of the azimutal velocity component is constant and differs from zero at large distances from the jet axis:

$$y \to \infty: \quad yw \to 2\pi\Gamma_0 = \text{const} \neq 0\,. \tag{5}$$

In other words, in stipulating (5) the distribution of w in the outer region corresponds to a free vortex ($w \sim 1/y$). The first problem, starting from the basic work of L. G. Loitsyanskii [1], is called a classical twisted jet; the second problem relates directly to modeling such intriguing natural phenomena as convective vortices, waterspouts, tornadoes, etc.

We embark on a solution of problem (1)-(4). Introducing the stream function ψ by the formulas $u = (1/y)(\partial\psi/\partial y)$, $v = (-1/y)(\partial\psi/\partial x)$ and passing over to the new variables

$$u = \left(\frac{g\beta Q_0}{4\pi\mu C_p}\right)^{1/2} f'(\eta)\,, \quad \Delta T = \frac{Q_0}{2\pi\mu C_p} h(\eta)\, x^{-1}\,,$$
$$w = \frac{L_0}{4\pi\mu}\left(\frac{g\beta Q_0}{\pi\mu C_p \nu^2}\right)^{1/4} b(\eta)\, x^{-3/2}\,, \quad \eta = \left(\frac{g\beta Q_0}{16\pi\mu C_p \nu^2}\right)^{1/2} \frac{y^2}{x}\,, \tag{6}$$

we obtain the basic equations in the form (a prime denotes a derivative with respect to η)

$$\eta f''' + f'' + \frac{1}{2} f f'' + h = 0\,, \quad \frac{1}{\mathrm{Pr}}(\eta h')' + \frac{1}{2} f h' + \frac{1}{2} f' h = 0\,,$$
$$\eta b'' + b' + \frac{1}{2} f b' + \frac{1}{2} f' b - \frac{1}{4\eta} b(1 - f) = 0\,, \tag{7}$$

which must be integrated under the following conditions:

$$\eta = 0: \quad f = \sqrt{\eta}\, f'' = \sqrt{\eta}\, h' = b = 0\,,$$
$$\eta = \infty: \quad f' = h = b = 0\,, \tag{8}$$

TABLE 1. Comparison of Values of $f(0, \mathrm{Pr})$

Pr	[7]	[8]	[9]	[3]	[11]	Present work
0.01	0.380	0.3715	–	0.3697	–	0.369494
0.03	–	0.55450	–	–	–	0.554503
0.05	–	–	–	0.6606	–	0.660663
0.1	–	0.82320	–	0.8232	–	0.823208
0.3	–	1.10774	–	–	–	1.107747
0.5	–	–	–	1.2409	–	1.240972
0.7	1.327	1.32629	–	–	1.3265	1.326290
0.72	–	–	–	–	–	1.333325
0.73	–	–	–	1.3370	–	1.336768
1	1.414	1.41421	1.4142	1.4146	–	1.414214
2	1.580	1.58113	1.5812	1.5811	–	1.581139
3	–	1.67973	–	–	–	1.679740
5	–	1.80667	1.8065	1.8068	–	1.806687
6.7	–	–	–	1.8808	–	1.880462
7	–	–	–	–	1.8908	1.891541
10	1.984	1.98184	1.9819	1.9825	–	1.981849
30	–	2.25559	–	–	–	2.255593
50	–	–	–	2.3773	2.3767	2.377851
100	–	2.53690	–	2.5368	2.5368	2.536915

$$\int_0^\infty f'h\,d\eta = 1\,,\quad \int_0^\infty f'b\eta^{1/2}d\eta = 1\,.$$

In the second case, assuming

$$w = \pi\Gamma_0\left(\frac{g\beta Q_0}{\pi\mu C_p\nu^2}\right)^{1/4} b(\eta)\,x^{-1/2}\,, \tag{9}$$

we have

$$\eta b'' + b' + \frac{1}{2}fb' - \frac{1}{4\eta}b(1-f) = 0\,, \tag{10}$$

where the boundary conditions are

$$b(0) = 0\,,\ \lim_{\eta\to\infty}\sqrt{\eta}\,b = 1\,. \tag{11}$$

It should be noted that in the notation of Eqs. (7) the term $\partial P/\partial x$ in the first equality of system (1) is omitted since the laws of diminution of the axial and azimuthal velocity components are different. In other words, in what follows, a self-similar flow mode occurring at some distance from the source is analyzed. From the physical point of view solutions (6), (9) reflect "elaboration" of geometrically similar profiles of velocity and temperature in space when $x \to \infty$, which is a common property for the jet process considered.

Results of Calculation. A specific feature of the obtained nonlinear system of equations is its singularity at the point $\eta = 0$, which makes it necessary to use as the initial value of the variable not zero, but a value close to it; in this case the boundary conditions should also be altered on the basis of expansion of the function into the

TABLE 2. Comparison of Values of $h(0, \mathrm{Pr})/\mathrm{Pr}$

Pr	[7]	[8]	[9]	[3]	[11]	Present work
0.01	0.759	0.8084	–	0.7994	–	0.798629
0.03	–	0.79193	–	–	–	0.791936
0.05	–	–	–	0.7859	–	0.785955
0.1	–	0.77287	–	0.7728	–	0.772883
0.3	–	0.73473	–	–	–	0.734731
0.5	–	–	–	0.7084	–	0.708459
0.7	0.687	0.68874	–	–	0.6894	0.688735
0.72	–	–	–	–	–	0.687022
0.73	–	–	–	0.6864	–	0.686184
1	0.667	0.66667	0.6666	0.6670	–	0.666667
2	0.625	0.62500	0.6252	0.6249	–	0.625000
3	–	0.60420	–	–	–	0.604196
5	–	0.58313	0.5806	0.5831	–	0.583135
6.7	–	–	–	0.5736	–	0.573489
7	–	–	–	–	0.5724	0.572181
10	0.561	0.56268	0.5630	0.5630	–	0.562680
30	–	0.54309	–	–	–	0.543087
50	–	–	–	0.5370	0.5376	0.537250
100	–	0.53133	–	0.5325	0.5318	0.531330

Taylor series (just this scheme is used in [7-11]). But this creates difficulties in numerical integration of the problem and leads to some loss of accuracy for the results obtained. An alternative approach that provides more reliable numerical data is suggested in [1]. It consists in replacement of the equations involving the singularity by an equivalent system not containing singularities. In our case this is performed by passing from the variable η to the variable ξ by the formula $\xi = \ln \eta$. Then, the boundary-value problem for different values of Pr was transformed to a Cauchy problem that was solved by the standard Runge–Kutta method. Missing initial conditions were determined by conjugating the iterative numerical solution via an initial boundary condition. To determine the accuracy of the computational scheme a series of test solutions were conducted for Prandtl numbers equal to 1 and 2. Good agreement between the results of the calculations and the previously known analytical solutions [2, 4, 12]

$$\mathrm{Pr} = 1: \; f(\infty) = 6, \; f'(0) = \sqrt{2}, \; h(0) = 2/3, \; d'(0) = \sqrt{2}/3;$$

$$\mathrm{Pr} = 2: \; f(\infty) = 4, \; f'(0) = \sqrt{5/2}, \; h(0) = 5/4,$$

$$\Gamma(\infty) = 0: \; d'(0) = \sqrt{45/128}, \; \Gamma(\infty) = \mathrm{const} \neq 0: \; d'(0) = \sqrt{5/32}$$

was a positive result of this test. Tables 1-4 present data on the basic hydrodynamic and thermal parameters of the jet process

$$\frac{m}{\mu} = 2\pi f(\infty)\, x, \quad \frac{ux}{\nu} = \frac{1}{2} f'(\eta)\, \mathrm{Gr}_x^{1/2},$$

$$\frac{g\beta \Delta T x^3}{\nu^3} = \frac{1}{2} h(\eta)\, \mathrm{Gr}_x, \quad \mathrm{Gr}_x = \frac{g\beta Q_0 x^2}{\pi \rho C_p \nu^3}$$

TABLE 3. Comparison of Values of $f(\infty, \mathrm{Pr})$

Pr	[7]	[8]	[9]	[3]	[10]	[14]	Present work
0.01	472.69	374.97	–	540	–	–	546.6
0.03	–	172.21	–	–	–	–	182.3
0.05	–	–	–	108.3	–	–	109.43
0.1	–	53.33	–	54.6	–	–	54.79
0.3	–	17.75	–	–	–	–	18.40
0.5	–	–	–	11.17	–	–	11.18
0.7	7.53	8.07	–	–	–	7.91	8.15
0.72	–	–	–	–	8.062	–	7.95
0.73	–	–	–	7.85	–	–	7.85
1	6.00	5.96	6.000	6.00	6.000	–	6.00
2	4.00	3.97	4.000	4.00	4.000	–	4.00
3	–	3.53	–	–	–	–	3.60
5	–	3.32	3.398	3.39	3.375	–	3.40
6.7	–	–	–	3.35	–	–	3.35
7	–	–	–	–	–	3.08	3.34
10	2.67	3.18	3.309	3.30	3.143	–	3.31
30	–	3.11	–	–	–	–	3.275
50	–	–	–	3.25	–	–	3.27
100	–	3.09	–	3.25	–	–	3.268

TABLE 4. Comparison of Values of $d'(0, \mathrm{Pr})$

Pr	$\Gamma(\infty)=0$			$\Gamma(\infty)=\mathrm{const}\neq 0$		
	[2]	[3]	Present work	[4]	[3]	Present work
0.01	–	0.09454	0.094487	–	0.17576	0.175765
0.03	–	–	0.143444	–	–	0.259048
0.05	–	0.17262	0.172433	–	0.30445	0.304708
0.1	–	0.21886	0.218915	–	0.37035	0.370194
0.3	–	–	0.313080	–	–	0.461441
0.5	–	0.36912	0.369858	–	0.48310	0.483763
0.7	–	–	0.414856	–	–	0.485239
0.72	–	–	0.418972	–	–	0.484766
0.73	–	0.42142	0.421010	–	0.48451	0.484498
1	0.4714045	0.47171	0.471405	0.4714045	0.47124	0.471405
2	0.5929271	0.59207	0.592927	0.3952847	0.39527	0.395285
3	–	–	0.631827	–	–	0.336492
5	–	0.61904	0.620267	–	0.27384	0.273623
6.7	–	0.59573	0.594426	–	0.24618	0.245997
7	–	–	0.589997	–	–	0.242346
10	–	0.55240	0.552077	–	0.21677	0.216606
30	–	–	0.450031	–	–	0.167442
50	–	0.41532	0.415394	–	0.15389	0.153444
100	–	0.37996	0.378529	–	0.13993	0.139210

As should be expected, with an increase in Pr the rate of rotational motion in the jet, which can be characterized by $d'(0)$, becomes weaker ($d(\eta) = \eta^{1/2} b(\eta)$). An important effect of buoyancy forces consists in a nonmonotonic dependence for $d'(0)$, namely, up to a certain Prandtl number the numerical value of $d'(0)$ grows, and then it decreases. The latter is typical of both modes of jet flow. At the same time there are substantial differences: in the case (4) the threshold Prandtl number lies within the interval (2; 5); and for the mode (5) within the interval (0.5; 0.72). This result is a direct consequence of the "competition" between the velocity and temperature fields in flows induced by heat sources. Therefore, additional studies were conducted that were directed at the determination of threshold values of Pr. It was found that for a twisted jet, in which $\Gamma = 0$, $d'(0)$ has a maximum equal to 0.634741 and the latter is attained at $\mathrm{Pr}_* = 3.475$:

$$f(\infty) = 3.519\,, \quad f'(0) = 1.715965\,, \quad h(0) = 2.076478\,.$$

If the jet generated by the point heat source is twisted so that the circulation of the azimuthal velocity component at its outer boundary is a constant value different from zero, then $d'(0) = 0.486172$ and $\mathrm{Pr}_* = 0.617$:

$$f(\infty) = 9.162\,, \quad f'(0) = 1.294563\,, \quad h(0) = 0.429622\,.$$

When $\mathrm{Pr} \gg \mathrm{Pr}_*$ or $\mathrm{Pr} \ll \mathrm{Pr}_*$, the situation changes: buoyancy forces suppress the effects of rotation, with this result being more substantial in the range of small Prandtl numbers. And, finally, as regards the details of the free-convective flow studied, we note that at $\mathrm{Pr}_* = 0.36048$, the relative profiles of the axial velocity and temperature coincide. At other values of Pr the profiles of ΔT are "wider" than u when $\mathrm{Pr} < \mathrm{Pr}_*$ and "narrower" when $\mathrm{Pr} > \mathrm{Pr}_*$. As is known [13], in a plane free-convective jet the ratio of the thicknesses of the dynamic and temperature layers is equal to unity at $\mathrm{Pr}_* = 5/9 \approx 0.55556$.

NOTATION

u, v, w, components of the velocity vector; x, y, φ, cylindrical coordinates; P, pressure; T, temperature; ν, μ, kinematic and dynamic viscosities; ρ, density; Pr, Prandtl number; C_p, heat capacity at constant pressure; g, acceleration of gravity; β, coefficient of volumetric thermal expansion; Gr_x, local Grashof number; $\Gamma = yw$, circulation of w.

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